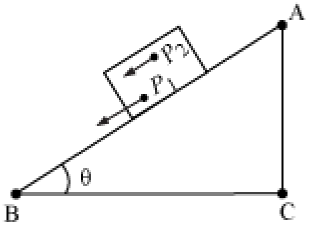
System Of Particles And Rotational Motion

Various Motions Possessed by a Rigid Body

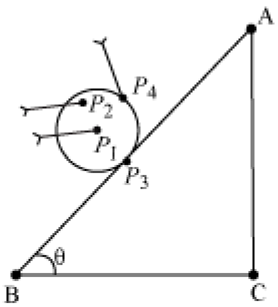
A rigid body possesses different types of motion.

For example:

- Translational motion
- Rotational motion
- Combination of translational and rotational motion
- **Translational Motion**



- The motion of the block sliding down an inclined plane is translational motion.
- All the particles of the body move together i.e, they have the same velocity at any instant of time.
- **Rotational Motion**



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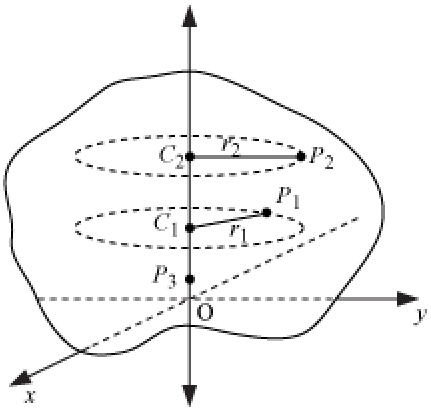
- If we fix the rigid body shown in the above figure along a straight line, then the body will undergo rotational motion.

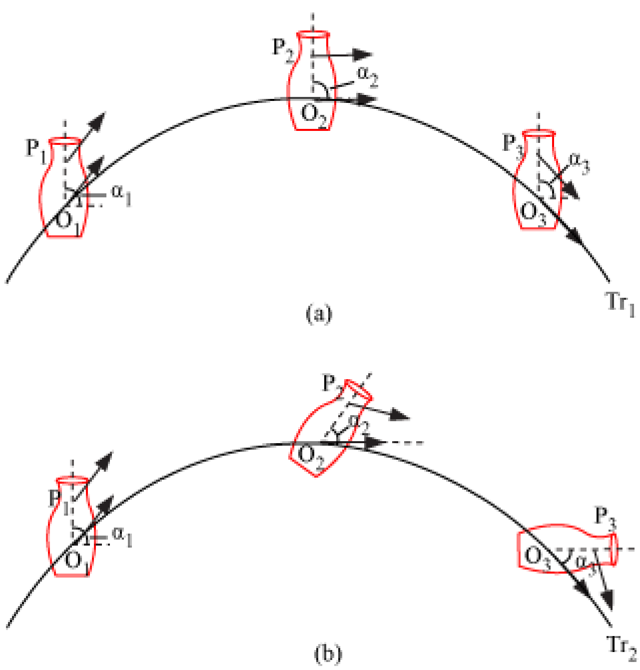
- The examples of rotational motion are motion of a ceiling fan, a potter's wheel, a merry go-round, etc.



In rotation of a rigid body about a fixed axis, every particle of the body moves in a circle, which lies in a plane perpendicular to the axis and has its centre on the axis.

- **Combination of translational and rotational motion**

The motion of a rigid body, which is not pivoted or fixed in some way, is either a pure translation or a combination of translation and rotation as shown in figure a & b.

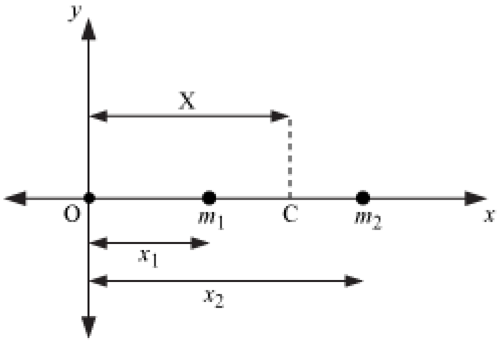


(a)

(b)

Centre of Mass

- **Centre of Mass of System of two Particles**



Let us consider a system of two particles having mass m_1 and m_2. Let the distances of two particles be x_1 and x_2 respectively from some origin O.

The centre of mass of the system is that point C, which is at a distance X from O, where X is given by,

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad \text{...(i)}$$

If $m_1 = m_2 = m$, then

$$X = \frac{mx_1 + mx_2}{2m} = \frac{x_1 + x_2}{2}$$

- **Centre of Mass of a System of *n* Particles**

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots + m_n x_n}{m_1 + m_2 + m_3 + \ldots + m_n}$$

$$\therefore \quad X = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

If the origin of the coordinate axes lies at the centre of mass, then $X = 0$

In such a case, the above equation becomes $\sum_i m_i x_i = 0$

- **Centre of Mass of a System of Three Particles**

Consider that three particles of masses m_1, m_2, and m_3 are lying at the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) respectively. Then, the centre of mass of the system of these three particles lies at a point, whose coordinates (X, Y) are given by,

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \quad \text{...(i)}$$

$$Y = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} \quad \text{...(ii)}$$

If $m_1 = m_2 = m_3 = m$, then

$$X = \frac{mx_1 + mx_2 + mx_3}{m + m + m} = \frac{x_1 + x_2 + x_3}{3}$$

$$Y = \frac{my_1 + my_2 + my_3}{m + m + m} = \frac{y_1 + y_2 + y_3}{3}$$

The centre of mass of the system of three particles coincides with the centroid of the triangle formed by the particles.

The results of equations (i) and (ii) can be generalized to a system of n particles distributed in space. The centre of mass of such a system is at (X, Y, Z), where

$$X = \frac{\sum m_i x_i}{M}$$

$$Y = \frac{\sum m_i y_i}{M}$$

$$Z = \frac{\sum m_i z_i}{M}$$

- **Centre of Mass of a Rigid body Having Continuous Distribution of Mass**

In order to find the coordinates of the centre of mass of a rigid body having continuous distribution of mass, we divide the body into n elementary parts of masses Δm_1, Δm_2, Δm_3, … , Δm_n. If x_1, x_2, x_3, … , x_n are x-coordinates of the various particles, then the X-coordinate of the centre of mass of the body is given by,

$$X = \frac{\Delta m_1 x_1 + \Delta m_2 x_2 + \Delta m_3 x_3 + \ldots + \Delta m_n x_n}{m_1 + m_2 + \ldots + m_n}$$

$$X = \frac{\sum_i \Delta m_i x_i}{\sum_i \Delta m_i}$$

For a rigid body having continuous distribution of mass, $n \rightarrow \infty$

Hence, $\Delta m_i \rightarrow 0$

$$\therefore X = \underset{\Delta m_i \rightarrow 0}{Lt} \frac{\sum_i \Delta m_i x_i}{\sum_i \Delta m_i} = \frac{\int x dm}{\int dm}$$

Hence, x is the distance of an elementary portion of mass dm of the body.

$\int dm = M$ = Total mass of the body

$$\therefore X = \frac{1}{M} \int x \, dm \qquad \ldots\text{(i)}$$

Similarly Y- and Z-coordinates of the centre of mass of the body are given by,

$$Y = \frac{1}{M} \int y \, dm \qquad \ldots\text{(ii)}$$

$$Z = \frac{1}{M} \int z \, dm \qquad \ldots\text{(iii)}$$

If the origin of the coordinate axis lies at the centre of mass, then

$X = Y = Z = 0$

In that case,

$$\int x dm = \int y dm = \int z dm = 0 \qquad \ldots\text{(iv)}$$

The equations (i), (ii), and (iii) are expressed in terms of the position vector. If $\vec{r}$ is the position vector of an elementary portion of mass dm of the body, then the position vector $\vec{R}$ of the centre of mass is given by,

$$\vec{R} = \frac{1}{M}\int \vec{r}\, dm$$

Motion of Centre of Mass and Linear Momentum of a System of particles

- Consider a system of n particles of masses m_1, m_2, m_3, ..., m_n. Suppose that $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_n$ are the position vectors of the n particles with respect to the origin of the coordinate axes. Then the position vector R→R→ of the centre of mass of the system is given by,

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_n}{m_1 + m_2 + m_3 + ...m_n}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_n}{M}$$

Where,

$M = m_1 + m_2 + m_3 + ... + m_n$ is the total mass of the system

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_n \qquad ...(i)$$

Differentiating both sides of equation (i) with respect to t, we obtain

$$M\frac{d\vec{R}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + ... + m_n\frac{d\vec{r}_n}{dt} \qquad ...(ii)$$

Then, $\frac{d\vec{R}}{dt} = \vec{V},$ the velocity of the centre of mass

And, $\frac{d\vec{r}_1}{dt} = \vec{V}_1, \frac{d\vec{r}_2}{dt} = \vec{V}_2, \frac{d\vec{r}_3}{dt} = \vec{V}_3, ... \frac{d\vec{r}_n}{dt} = \vec{V}_n$ are the velocities of the individual particles of the system

- Consider a system of n particles of masses m_1, m_2, m_3, ..., m_n. Suppose that $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_n$ are the position vectors of the n particles with respect to the origin of the coordinate axes. Then the position vector R→R→ of the centre of mass of the system is given by,

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_n}{m_1 + m_2 + m_3 + ...m_n}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_n}{M}$$

Where,

$M = m_1 + m_2 + m_3 + \ldots + m_n$ is the total mass of the system

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots + m_n\vec{r}_n \qquad \ldots(i)$$

Differentiating both sides of equation (i) with respect to *t*, we obtain

$$M\frac{d\vec{R}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + m_3\frac{d\vec{r}_3}{dt} + \ldots + m_n\frac{d\vec{r}_n}{dt} \qquad \ldots(ii)$$

Then, $\frac{d\vec{R}}{dt} = \vec{V}$, the velocity of the centre of mass

And, $\frac{d\vec{r}_1}{dt} = \vec{V}_1, \frac{d\vec{r}_2}{dt} = \vec{V}_2, \frac{d\vec{r}_3}{dt} = \vec{V}_3, \ldots \frac{d\vec{r}_n}{dt} = \vec{V}_n$ are the velocities of the individual particles of the system

$$\therefore M\vec{V} = m_1\vec{V}_1 + m_2\vec{V}_2 + m_3\vec{V}_3 + \ldots\ldots + m_n\vec{V}_n \quad \ldots (iii)$$

Again differentiating both sides of equation (iii) with respect to *t*, we obtain

$$M\frac{d\vec{V}}{dt} = m_1\frac{d\vec{V}_1}{dt} + m_2\frac{d\vec{V}_2}{dt} + m_3\frac{d\vec{V}_3}{dt} + \ldots + m_n\frac{d\vec{V}_n}{dt} \quad \ldots (iv)$$

Then, $\frac{d\vec{V}}{dt} = \vec{A}$, the acceleration of the centre of mass

And, $\frac{d\vec{V}_1}{dt} = \vec{a}_1, \frac{d\vec{V}_2}{dt} = \vec{a}_2, \ldots, \frac{d\vec{V}_n}{dt} = \vec{a}_n$ are the accelerations of the individual particles of the system

Substituting these values in equation (iv), we obtain

$$M\vec{A} = m_1\vec{a_1} + m_2\vec{a_2} + m_3\vec{a_3} + \ldots\ldots + m_n\vec{a_n}$$
$$M\vec{A} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \ldots\ldots\ldots + \vec{F_n}$$

Where, $\vec{F_1} = m_1\vec{a_1}, \vec{F_2} = m_2\vec{a_2}, \vec{F_3} = m_3\vec{a_3} \ldots\ldots$

$\vec{F}_n = m_n \vec{a}_n$ are the forces acting on the individual particles of the system

If $\vec{F}$ is the total force on the system, then

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n \qquad \ldots\text{(vi)}$$

From equations (v) and (vi), we obtain

$$\vec{F} = M\vec{A} \qquad \ldots\text{(vii)}$$

The force $\vec{F}$ includes the external forces $\vec{F}_{\text{ext}}$ and internal forces $\vec{F}_{\text{int}}$. Thus,

$$\vec{F} = \vec{F}_{ext} + \vec{F}_{\text{int}} \qquad \ldots\text{(vii)}$$

The internal forces between the individual particles of a system cancel out, provided the forces are central in nature.

$$\because \vec{F}_{\text{int}} = 0$$
$$\therefore \vec{F} = \vec{F}_{ext}$$
$$\boxed{M\vec{A} = \vec{F}_{ext}}$$

This implies that the centre of mass of a system of particles moves as if all the mass of the system is concentrated at the centre of the mass and all the external forces acting on the system are applied directly at this point.

Conservation of Linear Momentum of a System of Particles

- Consider a system of *n* particles of masses m_1, m_2, m_3 …m_n moving with respective linear velocities $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots \vec{v}_n$.

Suppose that a total external force $\vec{F}_{ext}$ acts on the system.

Total linear momentum of the system of *n*-particles,

$$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + \ldots\ldots + m_n \vec{v}_n$$

According to Newton's second law of motion,

$$\vec{F}_{\text{ext}} = \frac{d\vec{P}}{dt} = \frac{d}{dt}\left(m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots\ldots + m_{\text{n}} \vec{v}_{\text{n}}\right)$$

$$\text{If, } \vec{F}_{\text{ext}} = 0$$

$$\frac{d}{dt}\left(m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots\ldots + m_{\text{n}} \vec{v}_{\text{n}}\right) = 0$$

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots\ldots + m_{\text{n}} \vec{v}_{\text{n}} = \text{constant}$$

The above equation describes the law of conservation of linear momentum for a system of *n* particles, which states that if no external force acts on a system, then the total linear momentum of the system remains constant.

Torque and Angular Momentum

- Torque – It is measured by the product of magnitude of force and perpendicular distance of the line of action of force from the axis of rotation. Moment of force or torque = Force × Perpendicular distance $\vec{\tau} = \vec{r} \times \vec{F} = rF \sin\theta \hat{n}$

  Where, θ is smaller angle between $\vec{r}$ and $\vec{F}$; $\hat{n}$ is unit vector along $\vec{\tau}$

- Torque is the rotational analogue of force.
- Dimension of moment of force is $ML^2 T^{-2}$.
- It is a vector quantity.
- The direction of $\vec{\tau}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$, and is determined by right-handed screw rule.

Angular Momentum of a Particle

- It is the rotational analogue of linear momentum.

- Consider a particle of mass m and linear momentum $\vec{p}$ at a position $\vec{r}$ relative to the origin O. The angular momentum L of the particle with respect to the origin O is defined to be

  $\vec{L} = \vec{r} \times \vec{p}$

  Magnitude of the angular momentum vector is

  $L = rp \sin\theta$

  Where, p is the magnitude of $\vec{p}$ and θ is the angle between $\vec{r}$ and $\vec{p}$

  $\vec{L} = \vec{r} \times \vec{p}$

  Differentiating with respect to time,

  $$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$
  $$\frac{d\vec{L}}{dt} = \frac{d\,\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\,\vec{p}}{dt} \qquad \text{...(i)}$$

  Then, the velocity of the particle is $v = \frac{d\vec{r}}{dt}$ and $p = m\vec{v}$

  Because of this, $\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0$

  ∴Equation (i) becomes

  $$\vec{r} \times \frac{d\,\vec{p}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \qquad \left[\because \frac{d\,\vec{p}}{dt} = \vec{F}\right]$$

  Hence, $\frac{d}{dt}(\vec{r} \times \vec{p}) = \vec{\tau}$

  $$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

Torque and Angular Momentum for a System of Particles

Let us consider a system of n particles with angular momenta $\vec{L_1}, \vec{L_2}, \vec{L_3}, \ldots \vec{L_n}$. The total angular momentum of a system is $\vec{L} = \vec{L_1} + \vec{L_2} + \vec{L_3} + \ldots + \vec{L_n} = \sum_{i=1}^{n} \vec{L_i}$

The angular momentum of the i^{th} particle is given by,

$$\vec{L_i} = \vec{r_i} \times \vec{p_i}$$

$$\therefore \vec{L} = \sum_{i=1}^{n} \vec{L_i} = \sum_{i=1}^{n} \vec{r_i} \times \vec{p_i}$$

Then,

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\sum \vec{L_i}\right) \qquad \ldots(i)$$

$$= \sum_{i=1}^{n} \frac{d\vec{L_i}}{dt} = \sum_{i=1}^{n} \vec{\tau_i} \qquad \ldots(ii)$$

Where, $\vec{\tau_i}$ is the torque acting on i^{th} particle

$$\vec{\tau_i} = \vec{r_i} \times \vec{F_i}$$

The total torque can be given by,

$$\vec{\tau} = \sum_{i=1}^{n} \vec{\tau_i} = \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i}$$

As $\vec{F_i} = \vec{F}_{ext} + \vec{F}_{int}$, therefore,

$\vec{\tau}_{ext} = \sum_{i=1}^{n} \vec{r_i} \times \vec{F}_{ext}$, and $\vec{\tau}_{int} = \sum_{i=1}^{n} \vec{r_i} \times \vec{F}_{int}$

According to Newton's third law, the forces between any two particles of the system are equal and opposite. Therefore, contribution of internal forces to the total torque on the system is zero.

That is, $\vec{\tau}_{int} = 0$

$$\vec{\tau} = \sum_{i=1}^{n} \vec{\tau}_i = \vec{\tau}_{ext}$$

From equation (i),

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{\tau}_{ext}}$$

Principle of Conservation of Angular Momentum

When no external torque acts on a system of particles, the total angular momentum of the system always remains a constant.

$$\vec{\tau} = \frac{d}{dt}(\vec{L})$$

When no external torque acts on the system, that is

$$\vec{\tau} = 0$$
$$\frac{d}{dt}(\vec{L}) = 0$$

Equilibrium of a Rigid Body

For a rigid body to be in mechanical equilibrium, two conditions have to be satisfied.

- The net external force or the vector sum of all the external forces acting on the body must be zero i.e.,

$$\vec{F} = \sum_i \vec{F}_i = 0$$

From Newton's second law,

$\vec{F} = m\vec{a} = 0$, for translational equilibrium

$$\therefore \vec{a} = 0$$

$$\Rightarrow \frac{d\vec{v}}{dt} = 0$$

$\therefore \vec{v} =$ Constant or zero

The following points can be inferred from the above equation.

(i) When a body is in translational equilibrium, it will be either at rest ($v = 0$) or in uniform motion.

(ii) The body will have zero linear acceleration,

if $F = -\frac{dU}{dr}$, where U is potential energy of the body

In translational equilibrium, $F = -\frac{dU}{dr} = 0$

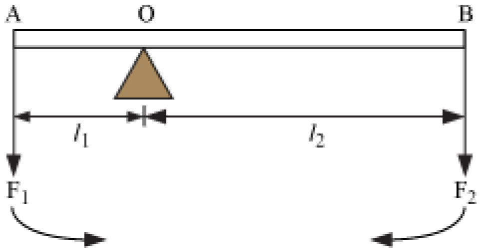
$\therefore U$ = constant

i.e., in equilibrium, potential energy of the body is constant (maximum or minimum).

- A rigid body is said to be in rotational equilibrium, if the body does not rotate or rotates with constant angular velocity. For this, the net external torque or the vector sum of all the torques acting on the body is zero.

  A body is in rotational equilibrium, when algebraic sum of moments of all the forces acting on the body about a fixed point is zero.

  Example – In case of beam balance or see-saw, the system will be in rotational equilibrium, if



$F_1 \times l_1 - F_2 \times l_2 = 0$

Now, $F_1 \times l_1 = +\vec{\tau}_1$ (anticlockwise moment)

And, $F_2 \times l_2 = -\vec{\tau}_2$ (clockwise moment)

$$\therefore \vec{\tau}_1 + \vec{\tau}_2 = 0$$
$$\sum \vec{\tau} = 0$$

i.e., for rotational equilibrium, total external force acting on the body must be zero.

The equation of motion of a rotating body is given by,

$$\tau_{\text{ext}} = \sum_{i=1}^{n} \vec{\tau}_i = \frac{\overrightarrow{dL}}{dt}$$

$\vec{L}$ = Constant

i.e., angular momentum of the body in rotational equilibrium will stay constant.

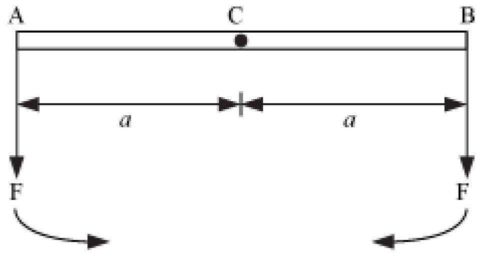
As $\vec{\tau}_{\text{ext}} = I\vec{\alpha} = 0,$
$\therefore \vec{\alpha} = 0$

i.e., angular acceleration of the body in rotational equilibrium will be zero.

Partial Equilibrium

A body is said to be in partial equilibrium if it is in translational equilibrium and not in rotational equilibrium or the body may be in rotational equilibrium and not in translational equilibrium.

Example – Let us consider a light rod AB of negligible mass with centre at C. Two parallel forces, each of magnitude *F*, are applied at the ends perpendicular to the rod as shown in the figure below.

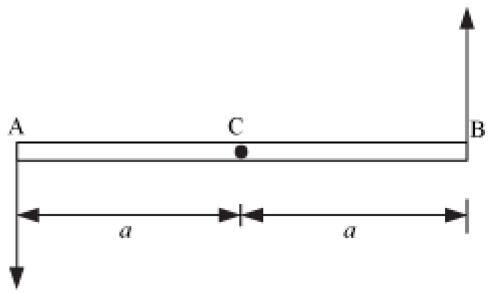


$F + F = 2F \neq 0$

As $\sum \vec{F} = 0$, therefore, the rod will not be in translational equilibrium.

The moment of force at A and B about fixed point C will be equal in magnitude (= *aF*), but opposite in sense. Therefore, the net moment of forces on the rod will be zero. Hence, the rod will be in rotational equilibrium.

Let the force applied at end B of the rod be reversed.



Here, total force on the rod = $F - F = 0$

∴ The rod is in translational equilibrium.

Moments of both forces about C are equal (= aF), but they are not opposite. They act in the same sense and cause anticlockwise rotation of the rod. Thus, the rod is not in rotational equilibrium.

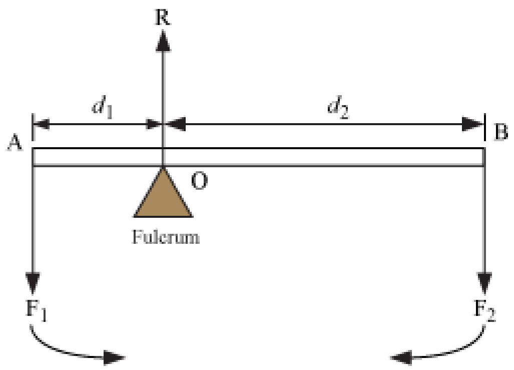
Important

A pair of equal and opposite forces with different lines of action is known as a couple or torque. A couple produces rotation without translation.

- **Principle Of Moments**

According to the principle of moments, a body will be in rotational equilibrium if algebraic sum of the moments of all forces acting on the body about a fixed point is zero.

Example − Take an ideal lever comprising of a light rod AB of negligible mass pivoted at a point O.



Here, F_1 and F_2 are two parallel forces. R is the reaction of the support at the fulcrum.

For translational equilibrium, net force should be equal to 0.

$\therefore R - F_1 - F_2 = 0$

$R = F_1 + F_2$ ... (i)

For rotational equilibrium, the algebraic sum of moments of forces about O must be zero. If AO = d_1 and OB = d_2, then

$F_1 \times d_1 - F_2 \times d_2 = 0$ ... (ii)

[Anticlockwise moments are taken as positive and clockwise as negative]

From equation (ii),

$F_1 d_1 = F_2 d_2$ ... (iii)

In case of the lever, force F_1 is usually some weight to be lifted (called load) and its distance from the fulcrum (AO = d_1) is called the load arm.

Force F_2 is the effort applied to lift the load and its distance from the fulcrum (OB = d_2) is called the effort arm.

From equation (iii), we have

Load × Load arm = Effort × Effort arm

The ratio F_1/F_2 is called mechanical advantage (M.A.) of the lever i.e.,

$$\text{M.A.} = \frac{\text{Load lifted}(F_1)}{\text{Effort applied}(F_2)} = \frac{d_2}{d_1}$$

Usually, M.A > 1 i.e., $F_1 > F_2$ i.e., a small effort is applied to lift a heavy load.

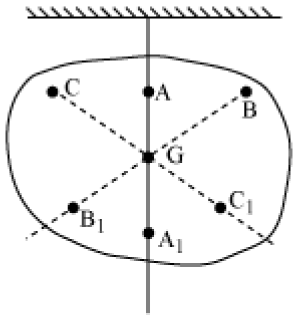
Centre of Gravity

- The centre of gravity (CG) of a body is a point where the weight of the body acts and total gravitational torque on the body is zero.
- The centre of gravity of the body coincides with the centre of mass of the body.

- If the body is so extended that $\vec{g}$ varies from part to part of the body, then the centre of gravity shall not coincide with the centre of mass of the body.

- The CG of a body of irregular shape can be determined by the following method.

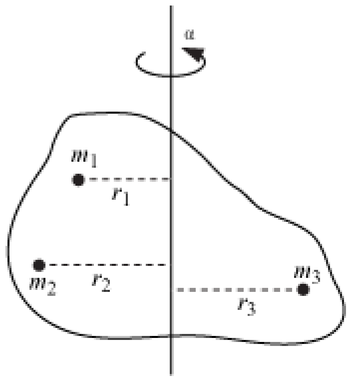
Suspend the body from some point such as A.



Draw the vertical line AA_1. It passes through CG.

Similarly, draw vertical lines BB_1 and CC_1 by suspending the body from some other points B and C, etc. The point of intersection G of these verticals gives us the position of CG of the irregular body.

Moment Of Inertia



- Moment of inertia of a body about a given axis is the sum of the products of masses of all the particles of the body and squares of their respective perpendicular distance from the axis of rotation.

$$I = \sum_{i=1}^{n} m_i r_1^2$$

- K.E. of rotation $= \frac{1}{2} I\omega^2$

  If ω =1, then

  K.E of rotation $= \frac{1}{2} I \times 1^2 = \frac{I}{2}$

  I = 2 × K.E. of rotation

  Thus, moment of inertia of a body about a given axis is equal to twice the K.E. of rotation of the body rotating with unit angular velocity about the given axis.

- **Physical significance of moment of inertia**

  We know that K.E. of rotation of a body $= \frac{1}{2} I\omega^2$

  Comparing it with K.E. of translation of the body $= \frac{1}{2} mv^2$,

  since linear velocity v is an analogue of angular velocity ω in rotational motion. Similarly, K.E. of translation is an analogue of K.E. of rotation in rotational motion. Therefore, mass (m) of the body is an analogue of moment of inertia (I) of the body in rotational motion.

- **How to calculate moment of inertia?**

  Example – Consider a thin ring of radius R and mass M, rotating in its own plane around its centre with angular velocity ω.

  Each mass element of the ring is at a distance R from the axis and moves with a speed $R\omega$.

  Therefore, the kinetic energy is

  $$K = \frac{1}{2} Mv^2 = \frac{1}{2} MR^2\omega^2 \qquad \text{...(i)}$$

  $$\because K = \frac{1}{2} I\omega^2 \qquad \text{...(ii)}$$

  Comparing equation (i) with equation (ii), we obtain

  $I = MR^2$, for the ring

- **Radius of gyration**

Radius of gyration of a body about a given axis is the perpendicular distance of a point P from the axis, where if whole mass of the body were concentrated, then the body shall have the same moment of inertia as it has with the actual distribution of mass. This distance is represented by *K*.

The momentum of inertia of a body of mass *M* and radius of gyration *K* is given by,

$I = MK^2$ ... (i)

Also, $I = m_1r_1^2 + m_2r_2^2 + \ldots + m_nr_n^2$

If $m_1 = m_2 = \ldots m_n = m$, then

$I = m(r_1^2 + r_2^2 + \ldots + r_n^2)$

$$\therefore I = m\left(\frac{r_1^2+r_2^2+r_3^2+\ldots.+r_n^2}{n}\right).n \text{ (Multiplying and dividing by } n\text{)}$$

However, $mn = M$ (mass of body)

$$\therefore I = M\left(\frac{r_1^2+r_2^2+r_3^2+\ldots.+r_n^2}{n}\right) \quad \ldots.\text{(ii)}$$

Comparing equations (i) and (ii), we obtain

$$MK^2 = M\left(\frac{r_1^2 + r_2^2 + \ldots + r_n^2}{n}\right)$$

$$\text{or } K^2 = \frac{r_1^2 + r_2^2 + \ldots + r_n^2}{n}$$

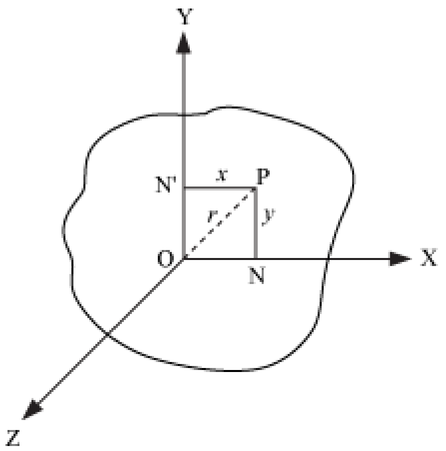
$$\text{or } K = \sqrt{\frac{r_1^2 + r_2^2 + r_n^2}{n}}$$

∴ The radius of gyration of a body about an axis is equal to the root mean square distance of the various particles constituting the body from the axis of rotation.

Theorems of Perpendicular and Parallel Axes

Theorem of Perpendicular Axis

The moment of inertia of a planar body about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.



Proof:
Consider a plane lamina lying in the XOY plane. The lamina is made up of a large number of particles. Consider a particle of mass *m* at P. Now, from P, drop perpendiculars PN' and PN on the X-axis and the Y-axis, respectively.

Now,
$PN' = x$ and PN = y

MI of the particle about the X-axis = my^2

MI of the whole lamina about the X-axis, $I_x = \sum my^2$

MI of the whole lamina about the Y-axis, $I_y = \sum mx^2$

MI of the whole lamina about the Z-axis, $I_z = \sum mr^2$

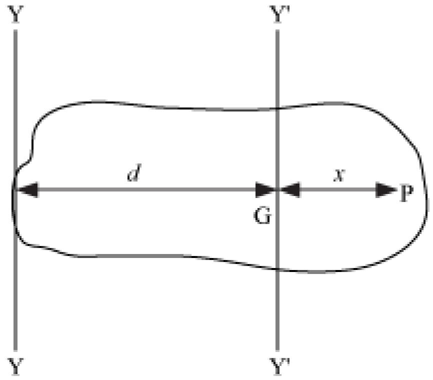
$r^2 = x^2 + y^2$

$\therefore I_z = \sum m\left(x^2 + y^2\right) = \sum mx^2 + \sum my^2$

Thus, we get:

$\boxed{I_z = I_x + I_y}$

Theorem of Parallel Axes



Statement: The moment of inertia of a body about any axis is equal to the sum of the moments of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.

Proof: Consider a particle of mass *m* at P. Let *d* be the perpendicular distance between parallel axes YY and YY' and let GP = *x*.

MI of the particle about YY = $m(x+d)^2$

M.I of the whole of lamina about YY,

$$I = \sum m(x+d)^2 = \sum m(x^2 + d^2 + 2xd)$$

$$\therefore I = \sum mx^2 + \sum md^2 + \sum 2mxd$$

Now,
$\sum mx^2 = I_g$ and $\sum md^2 = (\sum m)d^2 = Md^2$

Here, $M(=\sum m)$ is the mass of the lamina.

Also,
$\sum 2mxd = 2d\sum mx$

$$\therefore I = I_g + Md^2 + 2d\sum mx \qquad \text{(i)}$$

The lamina will balance itself about its centre of gravity. Therefore, the algebraic sum of the moments of the weights of constituent particles about the centre of gravity G should be zero.

$$\therefore \sum mg \times x = 0$$

$$g\sum mx = 0$$

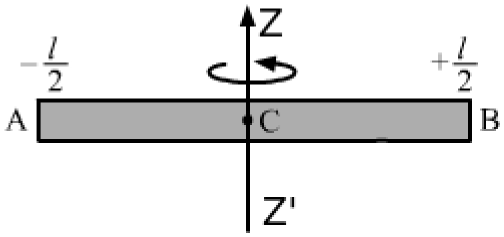
$$\sum mx = 0 \qquad [\because g \neq 0]$$

From equation (i), we get:

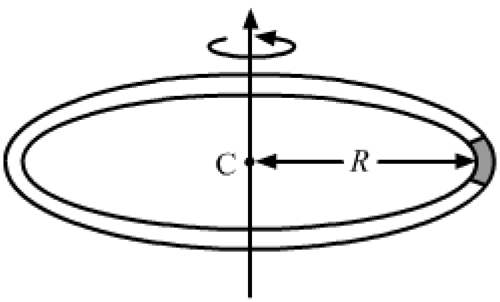
$$\boxed{I = I_g + Md^2}$$

APPLICATIONS:

We know that the moment of inertia of a thin uniform rod of mass *M* and length *l* about an axis passing through the centre of the mass of rod and perpendicular to its length is given by $I_C = \frac{Ml^2}{12}$.

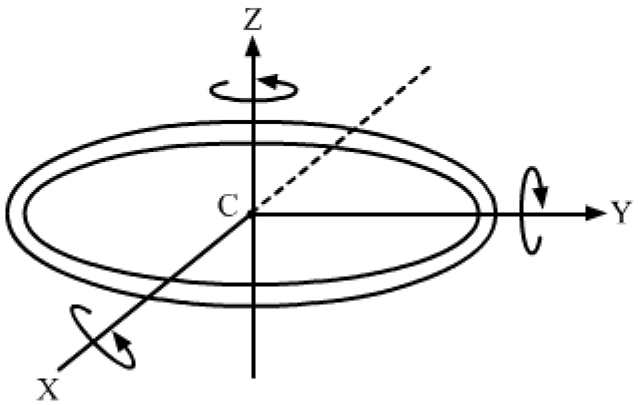


The moment of inertia of a thin ring of mass *M* and radius *R* about a transverse (perpendicular) axis passing through its centre is given by $I = MR^2$.



(a) Moment of inertia of a ring about diameter

Because a uniform ring has a symmetric shape, the moment of inertia of the ring about any diameter will be the same.

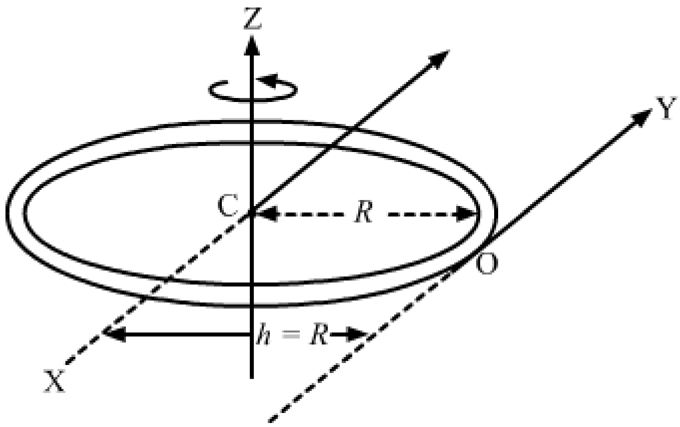


Let:
Z = Perpendicular axis through the centre of mass
$I_Z = I_C$ = Moment of inertia of the ring about Z
X and Y = Diameters
I_X and I_Y = Moments of inertia of the ring about X and Y, respectively.

Applying the perpendicular axis theorem, we get:
$I_Z = I_X + I_Y$ ...(1)
$I_Z = I_C$ and for the ring, $I_X = I_Y$.
Thus, equation (1) becomes $I_C = 2I_X = 2I_Y$.
Let $I_X = I_Y = I_D$, where I_D is the moment of inertia of the ring about the diameter.

$$\therefore I_C = 2I_D$$
$$\Rightarrow I_D = \frac{1}{2}I_C$$
$$\because I_C = MR^2$$
$$\Rightarrow I_D = \frac{1}{2}MR^2$$

(b) Moment of inertia of a ring about a tangent in its plane.



Let:
X = Diameter of the ring
Y = Tangent through point O parallel to X
I_O = Moment of inertia of the ring about Y
I_C = I_D = Moment of inertia of the ring X
R = Distance between X and Y.
∴h = R

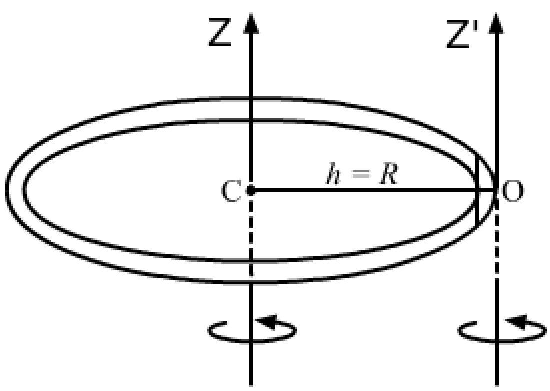
Applying the parallel axis theorem, we get:

$$I_O = I_C + Mh^2 \quad ...(1)$$

$$I_C = I_D = \frac{1}{2}MR^2 \text{ and } h = R$$

Thus, equation (1) becomes $I_O = \frac{1}{2}MR^2 + MR^2 = \frac{3}{2}MR^2$.

(c) Moment of inertia of a ring about a tangent perpendicular to its plane



Let:
I_O = Moment of inertia of the ring about the tangential axis passing through O
I_C = Moment of inertia of the ring about the axis passing through centre C of the ring
h = Perpendicular distance between the two parallel axes

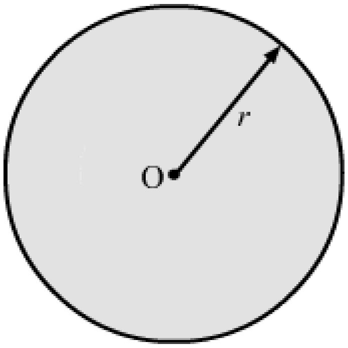
From the figure, we have:
$h = R$
Applying the parallel axis theorem, we get:

$$I_O = I_C + Mh^2 \quad ...(1)$$

$$I_C = MR^2 \text{ and } h = R$$

Thus, equation (1) becomes $I_O = MR^2 + MR^2 = 2MR^2$.

(iii) Moment of inertia of a disc about an axis passing through its centre and perpendicular to its plane



Consider a thin uniform disc of mass *m* and radius *r* rotating about an axis passing through its centre C and perpendicular to its plane, as shown in the figure.
Moment of inertia of the disc is given by

$I_C = \frac{1}{2}mr^2$

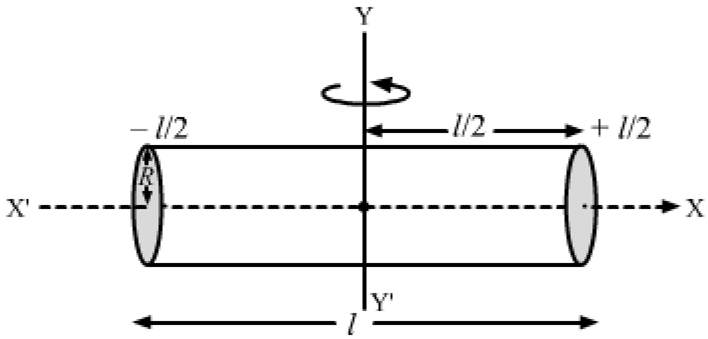
By applying the perpendicular axis theorem, we get the moment of inertia about the diameter of the disc as $I = \frac{1}{4}mr^2$. By applying the parallel axis theorem, we get the moment of inertia of the disc about the tangent in its plane as $\frac{5}{4}mr^2$ and the moment of inertia of the disc about the tangent perpendicular to the plane of the disc as $I = \frac{3}{2}mr^2$.

(iv) Moment of inertia of a solid cylinder about its geometrical axis

We can take a solid cylinder as a combination of a number of thin circular discs piled one over the other. Therefore, the moment of inertia of the solid cylinder about its geometrical axis will be the same as that of the disc.

$I_C = \frac{1}{2}MR^2$

(v) Moment of inertia of a solid cylinder about a transverse (perpendicular axis) passing through its centre



Let:
M = Mass of the cylinder
l = Length of the cylinder
R = Radius
XX' = Axis passing through the centre of mass
YY' = Axis passing through centre C and perpendicular to its own axis XX'
Moment of inertia of the cylinder about axis YY' passing through centre C and perpendicular to XX' is given by

$$I = \frac{MR^2}{4} + \frac{Ml^2}{12}$$

Kinematics of Rotational Motion About a Fixed Axis

Suppose a rigid body is undergoing circular motion about a given axis with a uniform angular acceleration α.

$$\alpha = \frac{d\omega}{dt}$$

$$d\omega = \alpha\, dt \qquad ...(i)$$

At $t = 0$, $\omega = \omega_0$

At $t = t$, $\omega = \omega$

Integrating (i) within proper limits, we obtain

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha dt$$

$$[\omega]_{\omega_0}^{\omega} = a[t]_0^t$$

$$\omega - \omega_0 = a\,(t-0) = \alpha t$$

$$\omega = \omega_0 + \alpha t$$

This is first kinematic equation.

If ω is angular velocity of the rigid body at any time t, then

$$\omega = \frac{d\theta}{dt}$$

$$\therefore d\theta = \omega dt \qquad \text{(ii)}$$

At $t = 0$, $\theta = 0$

At $t = t$, $\theta = \theta$

∴ Integrating (ii) within proper limits, we obtain

$$\int_0^{\theta} d\theta = \int_0^t \omega dt = \int_0^t (\omega_0 + \alpha t) dt$$

$$\int_0^{\theta} d\theta = \int_0^t \omega_0 dt + \alpha \int_0^t t dt$$

$$[\theta]_0^{\theta} = \omega_0 [t]_0^t + \alpha \left[\frac{t^2}{2}\right]_0^t$$

$$\theta - 0 = \omega_0 (t - 0) + \frac{\alpha}{2}(t^2 - 0)$$

$$\boxed{\theta = \omega_0 t + \frac{1}{2}\alpha t^2} \qquad \text{(iii)}$$

This is second kinematic equation.

We know that,

$$\omega = \frac{d\theta}{dt}$$

And, $\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\left(\frac{d\theta}{dt}\right)$

∴ $\alpha = \left(\frac{d\omega}{d\theta}\right)\omega$

$\omega\, d\omega = \alpha\, d\theta$ (iv)

When $\theta = 0, \omega = \omega_0$ and when $\theta = \theta, \omega = \omega$

∴ Integrating (iv) within proper limits, we obtain

$$\int_{\omega_0}^{\omega} d\omega = \int_0^{\theta} \alpha\, d\theta$$

$$\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha[\theta]_0^{\theta}$$

$$\frac{\omega^2 - \omega_0^2}{2} = \alpha(\theta - 0) = \alpha\theta$$

$$\boxed{\omega^2 - \omega_0^2 = 2\alpha\theta}$$

This is third kinematic equation.

Dynamics of Rotational Motion About a Fixed Axis

- In case of rotational motion about a fixed axis, two important points should be kept in mind.
- Only those forces that lie in planes perpendicular to the axis should be considered because the parallel forces will give rise to torques perpendicular to the axis of rotation, which will tend to turn the axis. As the axis is fixed, we will ignore the torques perpendicular to the axis.
- Only those components of position vectors should be considered which are perpendicular to the axis of rotation.
- Now, the small amount of work done (dW_1) by torque τ_1 in turning the body through a small angle $d\theta$ is given by,

  $dW_1 = \tau_1\, d\theta$ ... (i)

If there are more than one forces acting on the body such as τ_1, τ_2, τ_3, ...etc., then total work done on the body is given by,

$dW = (\tau_1 + \tau_2 + \tau_3 + \ldots)d\theta$ ... (ii)

As all the torques are parallel to the fixed axis of rotation, the magnitude τ of the total torque is the algebraic sum of the magnitudes of component torques i.e.,

$\tau = \tau_1 + \tau_2 + \tau_3 + \ldots$ ... (iii)

$\therefore dW = \tau\, d\theta$ ... ... (iv)

The corresponding relation for linear motion is

$dW = Fdx$ ... (v)

Dividing both sides of equation (iv) by time dt, we obtain

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

$P = \tau\omega$ ... (vi)

This is the instantaneous power associated with torque. This relation corresponds to expression for power in linear motion i.e.,

$P = Fv$ ... (vii)

In a perfectly rigid body, as there is no internal motion amongst the particles, therefore, there is no dissipation of energy. The rate at which work is done in the body is equal to the rate at which K.E. of the body increases i.e.,

$$\frac{dW}{dt} = \tau\omega = \frac{d}{dt}\left(\frac{1}{2}I\omega^2\right)$$

As I does not change with time,

$$\frac{dW}{dt} = \tau\omega = \frac{1}{2}I(2\omega)\frac{d\omega}{dt}$$

$$\tau\omega = \frac{dW}{dt} = I\omega\alpha \qquad \ldots\ldots \text{(viii)}$$

$\tau = I\ \alpha$ ... (ix)

Equation (ix) corresponds to equation of linear motion,

F = *ma*

Angular Momentum in Case of Rotation About a Fixed Axis

General expression for the total angular momentum of a system of particles:

$$\vec{L} = \sum_{i=1}^{n} \vec{r_i} \times \vec{p_i}$$

$$= \sum_{i=1}^{n} \vec{r_i} \times m_i\vec{v_i}$$

$$= \sum_{i=1}^{n} \vec{r_i} \times m_i(\vec{\omega} \times \vec{r_i})$$

From the vector triple product, we obtain:

$$\vec{L} = \sum_{i=1}^{n} m_i[(\vec{r_i} \cdot \vec{r_i})\vec{\omega} - (\vec{r_i} \cdot \vec{\omega})\vec{r}]$$

$$= \sum_{i=1}^{n} m_i[r_i^2\vec{\omega} - 0]$$

$$\vec{L} = \sum_{i=1}^{n} m_i r_i^2 \times \vec{\omega} \qquad \ldots\text{(i)}$$

For any particle, the angular momentum vector and the angular velocity vector are not necessarily parallel.

$\sum_{i=1}^{n} m_i r_i^2 = I$ (Moment of inertia of the body about the axis of rotation, which is here taken as Z-axis)

From (i), we get:

$\vec{L} = \vec{L}_z = I\omega\hat{k}$ ...(ii)

Differentiating both sides with respect to time, we get:

$\frac{d}{dt}\left(\vec{L}_z\right) = \frac{d}{dt}(I\omega)\hat{k}$ ($\hat{k}$ is a fixed constant vector) ... (iii)

For rotation about a fixed axis,

$\frac{d\vec{L}_z}{dt} = \tau\hat{k}$ ...(iv)

From (iii) and (iv), we get:

$\frac{d}{dt}(I\omega) = \tau$ ...(v)

If moment of inertia *I* does not change with time, then

$\boxed{I\frac{d\omega}{dt} = \tau}$

$\boxed{I\alpha = \tau}$

Principle of Conservation of Angular Momentum

According to this principle, the angular momentum of a system remains conserved if the net external torque on it is zero.

Proof:

Let us take a system of *n* particles that is rotating about a fixed axis. Now, let the angular momentum of the

system be $\overrightarrow{L}$.

The torque on the system is given by $\overrightarrow{\tau} = \frac{d\overrightarrow{L}}{dt}$.

As the net external torque on the system is zero, $\frac{d\overrightarrow{L}}{dt} = 0$. Thus, $\overrightarrow{L}$ is constant.

If $\overrightarrow{L}_1$, $\overrightarrow{L}_2$, $\overrightarrow{L}_3$,..., $\overrightarrow{L}_n$ are the angular momenta of the particles of the given system, then $\overrightarrow{L}_1 + \overrightarrow{L}_2 + \overrightarrow{L}_3 + ... + \overrightarrow{L}_n$ = constant.

As angular momentum is a vector quantity, its magnitude and direction will not change if there is no external torque on the system.

Some Applications of Conservation of Angular Momentum

Angular momentum is also represented by $\vec{L} = I\vec{\omega}$.

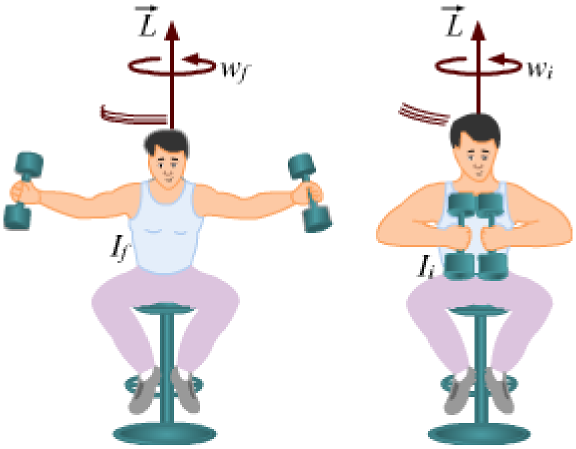
In other words, angular momentum depends on two quantities: moment of inertia I and angular velocity ω. For an isolated system, if there is any change in either of the quantities, the other quantity changes in order to keep their product (angular momentum) constant.

Some of the common applications of conservation of angular momentum are:

1. Increase in the angular velocity of a planet around the Sun as it comes near to it

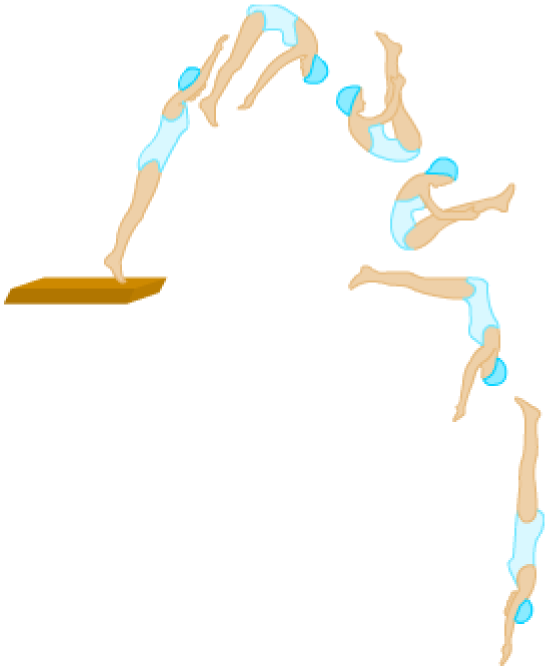
A planet revolves in an elliptical orbit, with the Sun at the focus. As the planet comes near the Sun, its moment of inertia decreases. As there is no external torque on the planet, its angular momentum remains conserved. For the conservation of the angular momentum of the planet, the speed of the planet increases as it comes near the Sun.

2. Change in the angular velocity of a person (sitting on a rotating chair) on folding of arms



A person is sitting on a rotating chair, with his arms outstretched. He is holding heavy weights in his hands. When he suddenly folds his arms, his angular velocity increases on account of decrease in the moment of inertia.

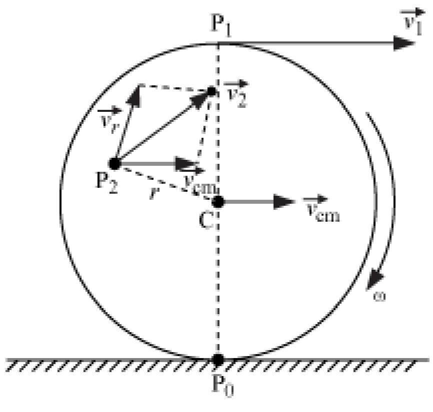
3. A diver jumping from a spring board performs somersaults in air



When the diver jumps from a springboard, he folds his body by rolling his hands and legs. By doing so, his moment of inertia decreases. His angular speed increases to keep the angular momentum constant. He then performs somersaults in the air. When he is about to reach the water's surface, he stretches his limbs and reduces his angular velocity.

Rolling Motion

- Rolling motion is a combination of translational motion and rotational motion.
- Let us consider the rolling motion of a circular disc on a level surface.



At any instant, the point of contact P_0 of the disc with the surface is at rest.

If $\vec{v}_{cm}$ is the velocity of centre of mass, then the translational velocity of the disc is $\vec{v}_{cm}$, which is parallel to the level surface.

Velocity $\vec{v}_2$ of any point P_2 of the disc is the vector sum of $\vec{v}_{cm}$ and $\vec{v}_r$, which is the linear velocity on account of rotation.

Magnitude of $\vec{v}_r$ is $r\omega$, where ω is the angular velocity of the rotation of the disc.

At point P_0, the linear velocity $\vec{v}_r$ due to rotation directed opposite to the translational velocity $\vec{v}_{cm}$ at P_0 is $R\omega$, where R is the radius of the disc.

P_0 is instantaneously at rest.

$$\therefore \vec{v}_{cm} = R\omega$$

Hence, for the disc to roll without slipping, the essential condition is $\boxed{\vec{v}_{cm} = R\omega}$.

Kinetic Energy of Rolling Motion

Total kinetic energy (K) = KE of translational motion of centre of mass (K_T) + KE of rotational motion about centre of mass (K_R)

$K = K_T + K_R$ ...(i)

If m is mass of the body and v_{cm} is the velocity of centre of mass of the body, then

$$K_T = \frac{1}{2}mv_{cm}^2$$

$$\text{and } K_R = \frac{1}{2}I\omega^2$$

Here, I is the moment of inertia of the body about the symmetry axis of the rolling body.

On putting the values in equation (i), we get:

$$K = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\omega^2 \qquad \text{... (ii)}$$

$v_{cm} = R\omega$

$\therefore \omega = \frac{v_{cm}}{R}$

Now,
I = mk^2
Here, k is the corresponding radius of gyration of the body.

Substituting the value in (ii), we obtain:

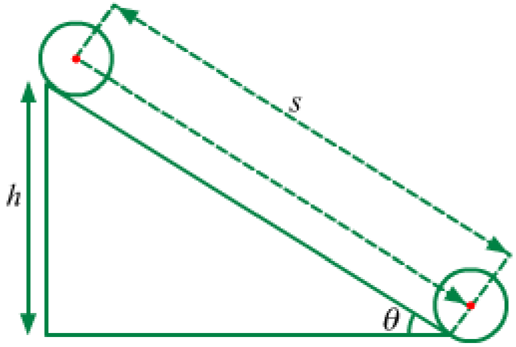
$$K = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}mk^2\left(\frac{v_{cm}}{R}\right)^2$$

$$\boxed{K = \frac{1}{2}mv_{cm}^2\left[1 + \frac{k^2}{R^2}\right]}$$

This is a general equation that applies to any rolling body—a disc, a ring, a cylinder or a sphere.

Rolling without slipping on an inclined plane

Let us take a rigid body of mass M and radius r rolling down an inclined plane of inclination θ and height h. As the body moves down the inclined plane, its gravitational potential energy is converted into kinetic energies of rotation as well as translation.



The body is at rest at the top of the inclined plane. Now, let the moment of inertia of the body be I, v be the linear speed and ω be the angular speed acquired by the body when it reaches the bottom of the inclined plane.

Loss in potential energy = Gain in kinetic energy
Loss in potential energy = Mgh

Gain in kinetic energy = $\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$ ...(1)

Let the radius of gyration of the body be *k*.

Moment of inertia can be written as *I* = *Mk*^{2}.

Relation between ω and *v* is given by $\omega = \frac{v}{r}$.

On replacing *I* and ω in equation (1), we get:

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}Mk^2\left(\frac{v}{r}\right)^2$$

$$\therefore v^2 = \frac{2gh}{1+\frac{k^2}{r^2}}$$

$$\Rightarrow v = \sqrt{\frac{2gh}{1+\frac{k^2}{r^2}}} \quad \ldots \left(2\right)$$

Now,

Let *a* be the linear acceleration of the body while rolling down the plane.

As the body starts from rest, initial velocity *u* is 0 and the length of the plane is *s.*

On applying $v^2 - u^2 = 2as$, we get:

$v^2 = 2as$

$\Rightarrow a = \frac{v^2}{2s}$

On substituting the value of v derived from equation (2), we get:

$$a = \frac{2gh}{\left(1+\frac{k^2}{r^2}\right)} \cdot \frac{1}{2s} \qquad \ldots \left(3\right)$$

$\sin\theta = \frac{h}{s}$

$\Rightarrow s = \frac{h}{\sin\theta}$

Thus, equation (3) becomes

$$a = \frac{2gh}{\left(1+\frac{k^2}{r^2}\right)} \times \frac{1}{2\left(\frac{h}{\sin\theta}\right)}$$

$$= \frac{g\sin\theta}{1+\frac{k^2}{r^2}}$$

Now, by substituting the value of radius of gyration (k) for different bodies, we can find out the velocity and acceleration of the rigid body rolling without slipping on the inclined plane.

The given table shows the values of v and a for rigid bodies of different shapes.

	Body	k	v	a
1	Ring or hollow cylinder	r	$\sqrt{gh}$	$\frac{1}{2}g\sin\theta$
2	Disc or solid cylinder	$\frac{r}{\sqrt{2}}$	$\sqrt{\frac{4}{3}gh}$	$\frac{2}{3}g\sin\theta$
3	Solid sphere	$\sqrt{\frac{2}{5}}r$	$\sqrt{\frac{10}{7}gh}$	$\frac{5}{7}g\sin\theta$